Final Exam — Analysis (WBMA012-05)

Monday 29 January 2024, 18.15h–20.15h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (15 points)

The so-called *diameter* of a nonempty and bounded set $A \subseteq \mathbb{R}$ is defined as

$$\operatorname{diam}(A) = \sup\{|x - y| \,:\, x, y \in A\}.$$

Prove that for the half-open interval $A = (-1, 5]$ we have $\operatorname{diam}(A) = 6$.

Hint: for arbitrary $x, y \in A$ you can assume without loss of generality that $y \leq x$.

Problem 2 (8 + 7 = 15 points)

Consider the following sequence:

$$x_{n+1} = \frac{1}{4 - x_n} \quad \text{with} \quad x_1 = 3.$$

(a) Show that $x_{n+1} < x_n$ and $x_n > 0$ for all $n \in \mathbb{N}$.

(b) Prove that the sequence (x_n) converges and compute $\lim x_n$.

Problem 3 (5 + 5 + 5 = 15 points)

For all $n \in \mathbb{N}$ we define the number a_n by truncating the number $\sqrt{2}$ after the n-th decimal place. For example, $a_1 = 1.4$ and $a_2 = 1.41$.

Consider the set $A = \{a_n \,:\, n \in \mathbb{N}\}$.

(a) Is A open?

(b) Is A closed?

(c) Is A compact?

Please turn over for problems 4, 5 and 6!

Problem 4 (9 + 6 = 15 points)

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove the following statements:

(a) f is uniformly continuous on $[1, \infty)$.

(b) f is uniformly continuous on $[0, 1]$.

Problem 5 (10 + 5 = 15 points)

Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ given by

$$f_n(x) = 1 + x + x^2 + \cdots + x^{n-1}.$$

(a) Let $0 < a < 1$. Show that (f_n) converges uniformly on the set $A = [-a, a]$.

(b) Show that (f_n) does *not* converge uniformly on the set $B = (-1, 1)$.

Problem 6 (8 + 7 = 15 points)

Consider the function $f : [0, 1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \neq 1, \\ 2 & \text{if } x = 1. \end{cases}$$

(a) Given $\varepsilon > 0$ construct a partition P_ε of $[0, 1]$ such that $U(f, P_\varepsilon) < 1 + \varepsilon$.

(b) Prove that f is integrable on $[0, 1]$ and $\int_0^1 f = 1$.

Please do not forget to fill out the online course evaluation!

End of test (90 points)

Solution of problem 1 (15 points)

Let $x, y \in A$ be arbitrary. Without loss of generality, we can assume that $y \leq x$ (otherwise we interchange the roles of x and y). Since $x \leq 5$ and $y > -1$, and thus $-y < 1$, we have

$$|x - y| = x - y = x + (-y) < 5 + 1 = 6.$$

This shows that $u = 6$ is an upper bound for the set $S = \{|x - y| \, : \, x, y \in A\}$.
(5 points)

To show that $u = 6$ is the least upper bound for A, we can follow (at least) two different approaches.

Method 1. Let $b \in \mathbb{R}$ be an arbitrary upper bound for the set S. Then for all $x, y \in A$ with $y \leq x$ we have

$$x - y \leq b.$$

(2 points)

In particular, for $x = 5$ and $y_n = -1 + 1/n$, where $n \in \mathbb{N}$, we have

$$6 - 1/n \leq b.$$

(4 points)

Taking the limit for $n \to \infty$, the Order Limit Theorem implies that $6 \leq b$. By definition, this shows that $u = 6$ is the least upper bound for S.
(4 points)

Method 2. Let $\varepsilon > 0$ be arbitrary. First consider the case $\varepsilon \leq 6$. Take $x = 5$ and $-1 < y < -1 + \varepsilon$. Then $x, y \in A$ with $y \leq x$.
(4 points)

We have

$$|x - y| = x - y = 5 - y > 5 + (1 - \varepsilon) = 6 - \varepsilon.$$

(4 points)

When $\varepsilon > 6$ we can simply take $x = y = 5$ which gives

$$|x - y| = 0 > 6 - \varepsilon.$$

(1 point)

Lemma 1.3.8 (characterization of least upper bounds) implies that $\text{diam}(A) = \sup(S) = 6$.
(1 point)

Solution of problem 2 (8 + 7 = 15 points)

(a) We have $x_2 = 1$ so $x_2 < x_1$.
(1 point)

Now assume that for some $n \in \mathbb{N}$ we have $x_{n+1} < x_n$, and thus $4 - x_{n+1} > 4 - x_n$. By taking reciprocals the inequality signs flip again and we get $x_{n+2} < x_{n+1}$. By induction it follows that the sequence (x_n) is strictly decreasing, as claimed.
(3 points)

We clearly have $x_1 > 0$.
(1 point)

Now assume that for some $n \in \mathbb{N}$ we have $x_n > 0$. Since we already know that (x_n) is a decreasing sequence, we also know that $x_n \leq x_1 = 3$. This gives $1 \leq 4 - x_n < 4$. By taking reciprocals, we get that $x_{n+1} > 1/4 > 0$. By induction it follows that the terms of the sequence (x_n) are strictly positive, as claimed.
(3 points)

(b) Since the sequence (x_n) is decreasing and bounded from below, the Monotone Convergence Theorem implies that the sequence converges.
(2 points)

Write $x = \lim x_n$. Then we have $x = 1/(4 - x)$, or, equivalently, $x^2 - 4x + 1 = 0$.
(2 points)

The solutions of this quadratic equation are $x = 2 \pm \sqrt{3}$.
(2 points)

Since $2+\sqrt{3} > 3$ and the sequence (x_n) is decreasing, we conclude that $\lim x_n = 2-\sqrt{3}$.
(1 point)

Solution of problem 3 (5 + 5 + 5 = 15 points)

(a) The element $a_1 = 1.4$ belongs to A. For any $\epsilon > 0$ the set $V_\epsilon(a_1) = (a_1 - \epsilon, a_1 + \epsilon)$ contains irrational numbers. Note that all the elements of A are rational numbers; indeed, for any $n \in \mathbb{N}$ the number $10^n a_n$ is an integer. Hence, the inclusion $V_\epsilon(a_1) \subseteq A$ cannot hold. We conclude that A is not open.
(5 points)

(b) The sequence (a_n) belongs to the set A. Note that $|a_n - \sqrt{2}| \leq 1/10^n \to 0$ and $a_n \neq \sqrt{2}$ for all $n \in \mathbb{N}$. We conclude that $\sqrt{2}$ is a limit point of the set A. Since $\sqrt{2}$ is irrational and A only contains rational numbers it follows that A does not contain all its limit points. We conclude that A is not closed.
(5 points)

(c) There are three methods to show that A is not compact.

Method 1. If A is compact, then A is both closed and bounded. However, In part (b) we concluded that A is not closed. Therefore, A is not compact.
(5 points)

Method 2. Consider the sequence (a_n) as defined in the problem. Then (a_n) is a sequence in the set A (that is, all terms of the sequence are contained in the set). Note that (a_n) is a convergent sequence with $\lim a_n = \sqrt{2}$. Therefore, every subsequence (a_{n_k}) is also convergent and $\lim a_{n_k} = \sqrt{2}$.
(2 points)

However, since A only contains rational numbers, we have that $\sqrt{2} \notin A$. This means that A does not satisfy the definition of a compact set (namely that every sequence in the set has a convergent subsequence of which the limit also belongs to the set). We conclude that A is not compact.
(3 points)

Method 3. Note that the sets $O_k = (1, a_{k+1})$, where $k \in \mathbb{N}$ form an open cover for A.
(3 points)

However, for all $n \in \mathbb{N}$ we have that $O_1 \cup \cdots \cup O_n = (1, a_{n+1})$ which only contains the points $a_1, \ldots, a_n$. Therefore, A cannot be covered with finitely many of the sets O_k. This shows that A is not compact.
(2 points)

Method 4 (definitely a harder approach). Note that $\sqrt{2} \notin A$ since all elements of A are rational numbers. In particular, it follows that $|a_n - \sqrt{2}| > 0$ for all $n \in \mathbb{N}$. Consider the following open intervals:

$$O_k = \left(a_k - \frac{|a_k - \sqrt{2}|}{2}, a_k + \frac{|a_k - \sqrt{2}|}{2}\right).$$

Since $a_k \in O_k$ we have that

$$A \subseteq \bigcup_{k=1}^{\infty} O_k,$$

which means that the sets O_k form an open cover for A. If A is compact, then finitely many of these sets also cover A. That is, there exists $n \in \mathbb{N}$ such that $A \subseteq O_1 \cup O_2 \cup \cdots \cup O_n$.
(3 points)

Consider the following positive number:

$$\epsilon_0 = \min\left\{\frac{|a_k - \sqrt{2}|}{2} \;:\; k = 1, \ldots, n\right\}.$$

Since $\lim a_n = \sqrt{2}$ there exists $N \in \mathbb{N}$ such that $|a_N - \sqrt{2}| < \epsilon_0$. Note that for all $k = 1, \ldots, n$ we have

$$\begin{aligned}
\left|a_N - a_k\right| &= \left|(a_N - \sqrt{2}) - (a_k - \sqrt{2})\right| \\
&\geq \left||a_N - \sqrt{2}| - |a_k - \sqrt{2}|\right| \qquad \text{(use that } \left||u| - |v|\right| \leq \left|u - v\right|\text{)} \\
&= |a_k - \sqrt{2}| - |a_N - \sqrt{2}| \\
&> |a_k - \sqrt{2}| - \frac{|a_k - \sqrt{2}|}{2} \\
&= \frac{|a_k - \sqrt{2}|}{2},
\end{aligned}$$

which implies that $a_N \notin O_k$. But this implies that the sets $O_1, \ldots, O_n$ do not cover the set A. From this contradiction we conclude that A cannot be compact.
(2 points)

Solution of problem 4 (9 + 6 = 15 points)

(a) The usual rules for differentiating functions gives

$$f'(x) = \sin(1/x) - \frac{\cos(1/x)}{x}.$$

Taking absolute values implies that for all $x \geq 1$ we have

$$|f'(x)| \leq |\sin(1/x)| + \frac{|\cos(1/x)|}{x} \leq 1 + \frac{1}{x} \leq 2.$$

(3 points)

For any $x, y \in [1, \infty)$ with $x \neq y$ the Mean Value Theorem implies that there exists a point c between x and y such that

$$f(x) - f(y) = f'(c)(x - y).$$

Taking absolute values gives

$$|f(x) - f(y)| = |f'(c)|\,|x - y| \leq 2|x - y|.$$

Clearly, this inequality still holds when $x = y$ (because then both sides are zero).
(3 points)

Let $\varepsilon > 0$ be arbitrary and $\delta = \varepsilon/2$. For all $x, y \in [1, \infty)$ we have

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| \leq 2|x - y| < 2\delta = \varepsilon,$$

which shows that f is uniformly continuous on $[1, \infty)$.
(3 points)

(b) Note that the function f is differentiable, and hence continuous, at each point $x \neq 0$.
(1 point)

Since $|f(x)| \leq |x|$ it follows that f is also continuous at $x = 0$. This can be shown in two different ways.

Method 1. Let (x_n) be any sequence such that $x_n \to 0$. Then

$$|f(x_n) - f(0)| = |f(x_n)| \leq |x_n| \to 0,$$

which implies that $f(x_n) \to f(0)$.
(3 points)

Method 2. Let $\epsilon > 0$ be arbitrary and set $\delta = \epsilon$. If $|x - 0| < \delta$, then

$$|f(x) - f(0)| = |f(x)| \leq |x| = |x - 0| < \delta = \epsilon.$$

(3 points)

This implies that the function f is continuous on the compact set $[0, 1]$. Now recall the theorem stating that continuous functions on compact sets are uniformly continuous.
(2 points)

Solution of problem 5 (10 + 5 = 15 points)

(a) *Method 1.* Note that we have that

$$f_n(x) = \frac{1}{1-x} - \frac{x^n}{1-x}.$$

This implies that for all $x \in (-1,1)$ the sequence (f_n) converges pointwise to the function $f(x) = 1/(1-x)$.
(3 points)

Let $0 < a < 1$. For all $x \in A = [-a, a]$ we have

$$|f_n(x) - f(x)| = \frac{|x|^n}{1-x} \leq \frac{a^n}{1-x} \leq \frac{a^n}{1-a}.$$

(3 points)

From here there are two ways to show the uniform convergence.

Alternative A. We have

$$0 \leq \lim \left(\sup_{x\in A} |f_n(x) - f(x)| \right) \leq \frac{1}{1-a} \lim a^n = 0,$$

which implies that (f_n) converges uniformly to f on A.
(4 points)

Alternative B. Pick $N \in \mathbb{N}$ such that

$$N > \frac{\log(a(1-\epsilon))}{\log(a)}$$

Note that $\log(a) < 0$. For all $n \geq N$ we have $n \log(a) \leq N \log(a)$ and thus

$$|f_n(x) - f(x)| \leq \frac{a^n}{1-a} = \frac{e^{n\log(a)}}{1-a} \leq \frac{e^{N\log(a)}}{1-a} < \epsilon \quad \text{for all} \quad x \in A = [-a, a].$$

By definition (f_n) converges uniformly to f on A.
(4 points)

Method 2. The functions f_n are the partial sums of the power series $\sum_{k=0}^{\infty} x^k$. The latter series has a radius of convergence equal to $R = 1$ (which easily follows from either the ratio test or root test).
(5 points)

From the general theory of power series it follows that the partial sums converge uniformly on all compact sets within the interval of convergence. Indeed, pick a number $b \in (a, 1)$. Since $b \in (-R, R) = (-1, 1)$, Theorem 6.5.1 implies that the power series converges absolutely for any x satisfying $|x| < b$. In particular, this is the case for $x = a$. Theorem 6.5.2 implies that the power series converges uniformly on $[-a, a]$.
(5 points)

Method 3. For all $x \in [-a, a]$ we have $|x^k| \leq a^k$. Since $0 < a < 1$ the series $\sum_{k=0}^{\infty} a^k$ is convergent. By Corollary 6.4.5 (the Weierstrass M-test) it follows that the series $\sum_{k=0}^{\infty} x^k$ converges uniformly on $[-a, a]$.
(8 points)

But the functions f_n are just the partial sums of this series. Therefore, the sequence (f_n) converges uniformly on $[-a, a]$.
(2 points)

(b) For all $x \in B = (-1, 1)$ we have

$$|f_n(x) - f(x)| = \frac{|x|^n}{1-x}$$

But the right hand side is an unbounded function on the set B.
(2 points)

In particular, it is not true that

$$\lim_{n\to\infty} \left(\sup_{x\in B} |f_n(x) - f(x)| \right) = 0,$$

which means that the sequence (f_n) does not converge uniformly on the set B.
(3 points)

Solution of problem 6 (8 + 7 = 15 points)

(a) If $0 < \varepsilon < 2$, then we take $P_\varepsilon = \{0, 1 - \varepsilon/2, 1\}$ and get

$$\begin{aligned} U(f, P_\varepsilon) &= (1 - \varepsilon/2) \sup\{f(x) : x \in [0, 1 - \varepsilon/2]\} + (1 - (1 - \varepsilon/2)) \sup\{f(x) : x \in [1 - \varepsilon/2, 1]\} \\ &= (1 - \varepsilon/2) + \varepsilon \\ &= 1 + \varepsilon/2 \\ &< 1 + \varepsilon. \end{aligned}$$

(6 points; also fine with nonstrict inequality, i.e., $U(f, P_\varepsilon) \leq 1 + \varepsilon$**)**

If $\varepsilon \geq 2$, then simply take $P_\varepsilon = \{0, 1\}$ to get

$$U(f, P_\varepsilon) = (1 - 0) \sup\{f(x) : x \in [0, 1]\} = 2 < 1 + \varepsilon.$$

(2 points)

(b) For all partitions $P = \{x_0 < x_1 < \cdots < x_n\}$ of $[0, 1]$ we have

$$\begin{aligned} L(f, P) &= \sum_{k=1}^{n} (x_k - x_{k-1}) \inf\{f(x) : x \in [x_{k-1}, x_k]\} \\ &= \sum_{k=1}^{n} (x_k - x_{k-1}) \\ &= 1. \end{aligned}$$

(3 points)

By part (a) it follows that for all $\varepsilon > 0$ there exists a partition P_ε of $[0, 1]$ such that $U(f, P_\varepsilon) < 1 + \varepsilon$, which gives

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon.$$

By Theorem 7.2.8 it follows that f is integrable on $[0, 1]$.
(2 points)

Finally, note that by definition we have

$$\int_0^1 f = \sup\{L(f, P) : P \in \mathcal{P}\} = 1.$$

(2 points)